WARTIKAR-II

Linear D.E. with Constant Co-efficients.

Equation of the form:-

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x)$$

where $a_0, a_1, a_2, \dots a_n$ are constants & $f(x)$ is a function of X only, can be considered as a linear DE with const. co-eff.

[NOTE: For a linear D.E. the eq. should not contain (1) the dependent var. (y) & its derivative in the root sign or in the denominator
(2) Dep. var (y) & its derivative of degree higher than one i.e. y^2, $(dy/dx)^2$ terms should not be present
(3) Terms with two der. multiplied or dep. var & der. multiplied should not be present.]

If the diff. operator D stands for d/dx then,

$$Dy = \frac{dy}{dx},$$

$$D^n y = \frac{d^n y}{dx^n},$$

$$D^{n-1} y = \frac{d^{n-1} y}{dx^{n-1}}.$$

$\therefore$ Linear DE with const. co-eff. can be expressed as:-

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots a_n y = f(x)$$

$$\therefore (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-1} + \ldots a_n) y = f(x)$$

$$\therefore f(D) y = f(x)$$

NOTE: $D^n cy = c D^n y$

$D(y+z) = Dy + Dz$

$(D-m_1)(D-m_2)y = (D-m_2)(D-m_1)y$

$= [D^2 - (m_1+m_2)D + m_1 m_2]\, y$

$D^m (D^n) y = D^{m+n} y.$

$\therefore (D^2+3D-4)y = (D+4)(D-1)y$

$= (D-1)(D+4)y$

$= \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y$

Linear D.E. of the type $f(D)y = 0$ i.e. $f(x) = 0$.

Consider a linear D.E of type $f(D)y$

i.e. $(D-m_1)(D-m_2) \ldots (D-m_n) y = 0$

where $m_1, m_2, \ldots m_n$ are roots of auxiliary eq. $f(D) = 0$, then the G.S. of the eq. $f(D)y = 0$ is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$$

Q4. Solve $(2D^2+5D-12)y=0$

Eq. is of the form

$f(D)y=0$

where auxillary eq is

$2D^2+5D-12=0$

i.e. $2D^2+8D-3D-12=0$

$2D(D+4)-3(D+4)=0$

$\therefore (D+4)(2D-3)=0 \therefore D=-4, 3/2$

Thus The roots of the auxillary eq. are $m_1=-4$, $m_2=3/2$

$\therefore$ G.S. is

$$y=C_1e^{\frac{3x}{2}}+C_2e^{-4x}$$

NATURE OF ROOTS OF AUXILLARY EQ. $f(D)=0$	General Solution of Eq. $f(D)y=0$.
1) Real & different $D=m_1, m_2, \ldots m_n$	$y=C_1e^{m_1x}+C_2e^{m_2x}+\ldots \ldots+C_ne^{m_nx}$
2) Real & multiple $D=m_1, m_1$ due to factor $(D-m_1)^2$ {Repeated twice}	$y=(C_1x+C_2)e^{m_1x}$
$D=m_1, m_1, \ldots m_n$	$y=(C_1x^{n-1}+C_2x^{n-2}+\ldots+C_n)e^{m_1x}$
3) Imaginary roots occur as conjugate pairs $m_1=\alpha\pm i\beta$, ~~$m_2=\alpha-i\beta$~~	$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$
4) Repeated imaginary roots $m_1=\alpha\pm i\beta$, $m_2=\alpha\pm i\beta$ ~~$m_3=\alpha+i\beta$, $m_4=\alpha-i\beta$~~ {Repeated twice}	$y=e^{\alpha x}\{(C_1x+C_2)\cos\beta x+(C_3x+C_4)\sin\beta x\}$

Q1. $(D+1)^3(D-4)y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ Auxillary eq. is

$(D+1)^3(D-4)=0$

$\therefore D=4,-1,-1,-1.$

$\therefore$ Roots are $m_1=4, m_2=-1$ repeated thrice

$\therefore$ G.S. is

$y=c_1e^{4x}+(c_2x^2+c_3x+c_4)e^{-x}$

Q2. $(D^2-2D+2)y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ Auxillary eq. is

$D^2-2D+2=0$

$\therefore D=\frac{2\pm\sqrt{4-8}}{2}=1\pm i$

$\therefore$ Roots are $m_1=1\pm i$

$\therefore$ G.S. is

$y=e^{1x}(c_1\cos x+c_2\sin x)$

Q3. $(D^2-2D+2)^2y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ Auxillary eq. is

$(D^2-2D+2)^2=0$

$\therefore D=\frac{2\pm\sqrt{4-8}}{2}, \frac{2\pm\sqrt{4-8}}{2}$

$D=1\pm i, 1\pm i$

$\therefore$ Roots are $m_1=1\pm i$; ~~m_2 $1\pm i$~~ repeated twic

$\therefore$ G.S. is

$y=e^{x}\{(c_1x+c_2)\cos x+(c_3x+c_4)\sin x$

Q4. $(D-3)(D-2)^3y=0$

Eq is of the form

$f(D)y=0$

$\therefore$ Auxillary eq. is

$(D-3)(D-2)^3=0$

$\therefore D=3,2,2,2$

$\therefore$ The roots are

$m_1=3$; $m_2=2$, repeated thrice

$\therefore$ G.S. is

$y=C_1e^{3x}+(C_2x^2+C_3x+C_4)e^{2x}$

Q5 $(D^3+7D^2+16D+10)y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ A.E. is

$D^3+7D^2+16D+10=0$

$(D+1)(D^2+6D+10)=0$

$\therefore D=-1;\ D=\dfrac{-6\pm\sqrt{36-40}}{2}$

$\therefore D=-1;\ D=-3\pm i$

$$\begin{array}{r} D^2+6D+10 \\ D+1\,\overline{)\,D^3+7D^2+16D+10} \\ D^3+D^2 \\ \hline 6D^2+16D \\ -6D^2\pm6D \\ \hline 10D+10 \end{array}$$

$\therefore$ The roots are

$m_1=-1$, $m_2=-3\pm i$

$\therefore$ G.S. is

$y=c_1e^{-x}+e^{-3x}(c_2\cos x+c_3\sin x)$

Q6. $\dfrac{d^4y}{dx^4}+4y=0$

$(D^4+4)y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ A.E. is

$D^4+4=0$

$\therefore \cancel{D=(D^2)^2-(2i)^2=0}$

$\cancel{(D^2-2i)(D^2+2i)=0}$

$(D^2+2-2D)(D^2+2+2D)=0$

$\therefore D^2-2D+2=0$; $D^2+2D+2=0$

$\therefore D=\dfrac{2\pm\sqrt{4-8}}{2}$; $D=\dfrac{-2\pm\sqrt{4-8}}{2}$

$=1\pm i$; $D=-1\pm i$

The roots are $m_1 = 1 \pm i$; $m_2 = -1 \pm i$

$\therefore$ G.S. is

$y = e^{x}(c_1 \cos x + c_2 \sin x) + e^{-x}(c_3 \cos x + c_4 \sin x)$

Eg. V - A

1. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

$\therefore (D^3 + 2D^2 + D)y = 0$

$\therefore$ Eq is of the form

$f(D)\,y = 0$

$\therefore$ A.E. is

$D^3 + 2D^2 + D = 0$

$\therefore D(D^2 + 2D + 1) = 0$

$\therefore D(D+1)^2 = 0$

$\therefore D = 0, -1, -1$

$\therefore$ Root are $m_1 = 0$; $m_2 = -1$ {Repeated twice)

$\therefore$ G.S. is

$y = c_1 e^{0x} + (c_2 x + c_3)e^{-x}$

$= c_1 + e^{-x}(c_2 x + c_3)$

2. $(D^4 + 2D^2 + 1)y = 0$

Eq. is of the form

$f(D)\,y = 0$

$\therefore$ A.E. is

$D^4 + 2D^2 + 1 = 0$

$\therefore (D^2+1)^2 = 0$

$\therefore (D^2+1)(D^2+1) = 0$

$\therefore D = \pm i, \pm i$

$\therefore$ Roots are $m_1 = \pm i$, Repeated twice

$\therefore$ G.S. is

$y = e^{0x}\{(c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x\}$

$\therefore y = (c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x$

$(D^3+D^2-2D+12$

3. $\cancel{(D^3+2D^2+1)}y=0$

Eq. is of the form

$f(D)\,y=0$

$\therefore$ A.E. is

$\cancel{D^3+2D^2+1=0}$

$D^3+D^2-2D+12=0$

$(D+3)(D^2-2D+4)=0$

$\therefore D=-3,\ \dfrac{2\pm\sqrt{4-16}}{2}$

$=-3,\ 1\pm i\sqrt{3}$

$\therefore$ Roots are

$m_1=-3;\ m_2=1\pm i\sqrt{3}$

$\therefore$ G.S. is

$y=c_1e^{-3x}+e^{x}(c_2\cos\sqrt{3}x+c_3\sin\sqrt{3}x)$

$$\begin{array}{l}
\qquad\quad D^2-2D+4 \\
D+3\,\big)\,D^3+D^2-2D+12 \\
\qquad\ \ -D^3\mp3D^2 \\
\qquad\qquad -2D^2-2D \\
\qquad\qquad -2D^2\mp6D \\
\qquad\qquad\qquad 4D+12 \\
\qquad\qquad\qquad 4D+12
\end{array}$$

4. $(2D^2-D-10)y=0$

Eq. is of the form

$f(D)\,y=0$

$\therefore$ A.E. is

$2D^2-D-10=0$

$\therefore D=\dfrac{1\pm\sqrt{1+80}}{4}=\dfrac{1\pm9}{4}=\dfrac{5}{2}$ or -2

$\therefore$ Roots are $m_1=5/2,\ m_2=-2$

$\therefore$ G.S. is

$y=c_1e^{\frac{5}{2}x}+c_2e^{-2x}$

5. $(D^6+6D^4+9D^2)y=0$

Eq. is of the form

$f(D)\,y=0$

$\therefore$ A.E. is

$D^6+6D^4+9D^2=0$

$\therefore D^2(D^4+6D^2+9)=0$

$\therefore D^2(D^4+3D^2+3D^2+9)=0$

$\therefore D^2(D^2+3)^2=0$

$\therefore D = 0, 0, \pm i\sqrt{3}, \pm i\sqrt{3}$

$\therefore$ Roots are

$m_1 = 0$, repeated twice

$m_2 = \pm i\sqrt{3}$ " "

G.S. is

$$y = (c_1x+c_2)e^{0x} + e^{0x}\{(c_3x+c_4)\cos\sqrt{3}x + (c_5x+c_6)\sin\sqrt{3}x\}$$

$$\therefore y = c_1x+c_2+(c_3x+c_4)\cos\sqrt{3}x+(c_5x+c_6)\sin\sqrt{3}x$$

6. $(D^2+25)y=0$

Eq. is of the form

$f(D)y=0$

$\therefore$ A.E. is

$D^2+25=0$

$\therefore D^2=-25$

$\therefore D = \pm\sqrt{-25} = \pm 5i$

$\therefore$ Roots are $m_1 = \pm 5i$

$\therefore$ G.S. is

$$y = e^{0x}(c_1\cos 5x + c_2\sin 5x)$$

$$\therefore y = c_1\cos 5x + c_2\sin 5x$$

Type II: Eq. of the form $f(D)y=f(x)$

– Particular Integral by general method:-

Consider a linear D.E $f(D)y=f(x)$

G.S. is $y = y_c + y_p$ where

y_c is the complimentary function

& y_p is the particular integral

y_c is obtained by soln. to eq. $f(D)y_c=0$ & contains arbitary constants

y_p is obtained by soln. to eq $f(D)y_p=f(x)$ & does not contain arbitary const.

$\therefore y_p = \frac{1}{f(D)}f(x)$ where $\frac{1}{f(D)}$ is the inverse operator.

NOTE: (1) $\frac{1}{D} f(x) = \int f(x)dx$

(2) $\frac{1}{D^n} f(x) = \int\int \dots \int_{n \text{ times}} f(x)(dx)^n$

Eg. $\frac{1}{D} x^2 = \int x^2 dx = \frac{x^3}{3}$

$\frac{1}{D^3} x^2 = \int\int\int x^2 (dx)^3 = \frac{x^5}{3 \times 4 \times 5} = \frac{x^5}{60}$

(2) $\frac{1}{f(D)} a f(x) = a \frac{1}{f(D)} f(x)$

(3) $\frac{1}{D-m} f(x) = e^{mx} \int e^{-mx} f(x) dx$

(4) $\frac{1}{D+m} f(x) = e^{-mx} \int e^{mx} f(x) dx$

Eg. $\frac{1}{D-2} e^{3x} = e^{2x} \int e^{-2x} e^{3x} dx$
$= e^{2x} \int e^x dx = e^{2x} \cdot e^x$
$= e^{3x}$

Methods to solve for y_p (P.I.):

(1) Method of factors:

Eg. Obtain the PI for $(D-2)(D-1)y = e^{4x}$

$\therefore y_p = \frac{1}{(D-2)} \left\{ \frac{1}{(D-1)} e^{4x} \right\}$

$= \frac{1}{(D-2)} \left\{ e^x \int e^{-x} \cdot e^{4x} dx \right\} = \frac{1}{(D-2)} \left\{ e^x \cdot \frac{e^{3x}}{3} \right\}$

$= \frac{1}{(D-2)} \frac{e^{4x}}{3} = \frac{1}{3} \cdot \frac{1}{(D-2)} e^{4x}$

$= \frac{1}{3} e^{2x} \int e^{-2x} e^{4x} dx = \frac{1}{3} \cdot e^{2x} \cdot \frac{e^{2x}}{2}$

$= \frac{1}{6} e^{4x}$

(2) Method of P.F.:-

Eg. $(D-2)(D-1)y = e^{4x}$

$\therefore y_p = \frac{1}{(D-2)} \cdot \frac{1}{(D-1)} e^{4x}$

$$\therefore y_p = \left\{ \frac{1}{D-2} - \frac{1}{D-1} \right\} e^{4x}$$

$$= \frac{1}{(D-2)} e^{4x} - \frac{1}{(D-1)} e^{4x}$$

$$= e^{2x} \int e^{-2x} e^{4x} dx - e^{x} \int e^{-x} e^{4x} dx$$

$$= e^{2x} \cdot \frac{e^{2x}}{2} - e^{x} \cdot \frac{e^{3x}}{3} = \frac{e^{4x}}{2} - \frac{e^{4x}}{3}$$

$$= \frac{e^{4x}}{6}.$$

Q1. Find the C.F. func. of $(D^2+4D+3)y = e^{x}$

C.F. is obtained by solving

$f(D) y_c = 0$

$\therefore (D^2+4D+3) y_c = 0$

~~is of the form~~ $\therefore$ A.E. is

$D^2+4D+3=0$

$D^2+D+3D+3=0$

$\therefore (D+3)(D+1) = 0$

$\therefore D = -1, -3$

$\therefore$ Roots are $m_1 = -1$, $m_2 = -3$

$\therefore$ C.F. is

$y_c = c_1 e^{-3x} + c_2 e^{-x}$.

Q2 Solve $\frac{d^2y}{dx^2} - y = 2+5x$.

$(D^2-1)y = 2+5x$

C.F. is obtained by solving

$f(D) y_c = 0$

$(D^2-1) y_c = 0$

$\therefore$ A.E. is

$D^2-1=0 \quad \therefore D^2 = 1 \quad \therefore D = +1, -1$

$\therefore$ Roots are $m_1 = 1$, $m_2 = -1$

$\therefore$ C.F. is

$y_c = c_1 e^{x} + c_2 e^{-x}$

P.I. is given by

$(D^2-1) y_p = 2+5x$

$$\therefore y_p = \frac{1}{(D^2-1)}(2+5x)$$

$$= \frac{1}{(D-1)(D+1)}(2+5x)$$

$$= \left[\frac{1}{2(D-1)} - \frac{1}{2(D+1)}\right](2+5x)$$

$$\therefore y_p = \frac{1}{2(D-1)}(2+5x) - \frac{1}{2(D+1)}(2+5x)$$

$$= \frac{1}{2} \cdot e^{x} \int e^{-x}(2+5x)\,dx - \frac{1}{2} e^{-x} \int e^{x}(2+5x)\,dx$$

$$= \cancel{\frac{e^x}{2}\left(\int 2e^{-x}\,dx\right.}$$

$$= \frac{e^x}{2}\left[(2+5x)\frac{e^{-x}}{-1} - 5e^{-x}\right] - \frac{e^{-x}}{2}\left[(2+5x)e^{x} - 5e^{x}\right]$$

$$= -1 - \frac{5x}{2} - \frac{5}{2} - 1 - \frac{5x}{2} + \frac{5}{2}$$

$$\therefore y_p = -2-5x$$

$\therefore$ GS is $y = y_c + y_p$

$$\therefore y = c_1 e^{x} + c_2 e^{-x} - 2 - 5x.$$

Type III: P.I. by formula Method:-

FORMULA	CONDITION
(1) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$	$f(a) \neq 0$
(2) $\frac{1}{f(D)} e^{ax} = \frac{x^r}{r!} \frac{e^{ax}}{\phi(a)}$ $f(D) = (D-a)^r$	$f(a) = 0$
(3) $\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)$	$f(-a^2) \neq 0$
(4) $\frac{1}{(D^2+a^2)^r} \sin(ax+b) = \frac{(-1)^r}{(2a)^r} \frac{x^r}{r!} \sin\left(ax+b+\frac{r\pi}{2}\right)$	$f(-a^2) = 0$

(4) $\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$ $f(-a^2)\neq 0$

(5) $\frac{1}{f(D^2+a^2)^r}\cos(ax+b) = \frac{(-1)^r}{(2a)^r}\cdot\frac{x^r}{r!}\cos\left(ax+b+\frac{r\pi}{2}\right)$ $f(-a^2)=0$

(6) $\frac{1}{f(D^2)}\sinh(ax+b) = \frac{1}{f(a^2)}\sinh(ax+b)$ $f(a^2)\neq 0$

(7) $\frac{1}{f(D^2)}\cosh(ax+b) = \frac{1}{f(a^2)}\cosh(ax+b)$ $f(a^2)\neq 0$

(8) $\frac{1}{f(D)}x^m = \{f(D)\}^{-1}x^m$ expand $\{f(D)\}^{-1}$ using

$(1+x)^{-1} = 1-x+x^2-x^3\ldots\ldots$

$(1-x)^{-1} = 1+x+x^2+x^3\ldots\ldots$

$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

(9) $\frac{1}{f(D)}e^{ax}V f(x) = e^{ax}\cdot\frac{1}{f(D+a)}V f(x)$

(10) $\frac{1}{f(D)}[xV f(x)] = \left\{x-\frac{f'(D)}{f(D)}\right\}\cdot\frac{1}{f(D)}\cdot V.$

I-2hr

Q1. P.I. of $(D^3-2D^2-5D+6)y = e^{4x}$

Eq. is of the form

$f(D)\,y = f(x)$

$\therefore$ P.I. is given by

$f(D)\,y_p = e^{4x}$

$\therefore y_p = \frac{1}{(D^3-2D^2-5D+6)}\cdot e^{4x}$

$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

$\therefore y_p = \frac{1}{(4^3-2\times4^2-5\times4+6)}\cdot e^{4x} = \frac{e^{4x}}{18}$

Q2. $(D^3-5D^2+8D-4)y = e^{2x}+2e^{x}+3e^{-x}+2$

Eq. is of the form

$f(D)\, y = f(x)$

$\therefore$ C.F. is obtained by

$f(D)\, y_c = 0$

A.E. is given by

$D^3-5D^2+8D-4=0$

$(D-1)(D^2-4D+4)=0$

$(D-1)(D-2)^2=0$

$\therefore D = 1,2,2$

$\therefore$ Roots are $m_1=1$, $m_2=2$, repeated twice.

$$\begin{array}{r} D^2-4D+4 \\ D-1\overline{)D^3-5D^2+8D-4} \\ -D^3 \mp D^2 \\ \hline -4D^2+8D \\ \mp 4D^2 \pm 4D \\ \hline 4D-4 \end{array}$$

$\therefore$ C.F. is

$y_c = c_1 e^{x} + (c_2 x + c_3) e^{2x}$.

P.I. is given by

$f(D)\, y_p = f(x)$

$\therefore y_p = \dfrac{1}{(D-1)(D-2)^2} \cdot e^{2x} + \dfrac{2}{(D-1)(D-2)^2} \cdot e^{x}$

$+ \dfrac{3}{(D-1)(D-2)^2} e^{-x} + \dfrac{2}{(D-1)(D-2)^2} \cdot e^{0x}$.

$\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax} \quad \{f(a) \neq 0\}$

$\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f^{r}(a)} \cdot \dfrac{x^r}{r!} \cdot e^{ax} \quad \{f(a)=0\}$

$\therefore y_p = \dfrac{1}{1} \dfrac{x^2}{2!} e^{2x} + \dfrac{2}{(-1)^2} \cdot \dfrac{x^1}{1!} e^{x}$

$+ \dfrac{3}{(-2)(-3)^2} e^{-x} + \dfrac{2}{(-1)(-2)^2}$

$y_p = \dfrac{x^2 e^{2x}}{2} + 2xe^{x} - \dfrac{e^{-x}}{6} - \dfrac{1}{2}$

$\therefore$ G.S. is $y = y_c + y_p$

$\therefore y = c_1 e^{x} + (c_2 x + c_3) e^{2x} + \dfrac{x^2 e^{2x}}{2} + 2xe^{x} - \dfrac{e^{-x}}{6} - \dfrac{1}{2}$

Q3 P.I. $(D^3+D^2+D+1)y = \sin 2x$

Eq. is of the form

$f(D)\, y = f(x)$

~~To obtain~~ P.I. is given by

$f(D)\, y_p = \sin 2x$

$\therefore y_p = \dfrac{1}{(D^3+D^2+D+1)} \sin 2x$

$\dfrac{1}{f(D^2)} \sin(ax+b) = \dfrac{1}{f(-a^2)} \sin(ax+b)$

$y_p = \dfrac{1}{(D.D^2+D^2+D+1)} \sin 2x$

$= \dfrac{1}{D(-4)+(-4)+D+1} \sin 2x$

$= \dfrac{1}{-3D-3} \sin 2x$

$= \dfrac{1}{-3} \cdot \dfrac{1}{D+1} \cdot \dfrac{D-1}{D-1} \sin 2x$

$= \dfrac{-1}{3}(D-1) \dfrac{1}{(D^2-1)} \sin 2x$

$\therefore y_p = -\dfrac{1}{3}(D-1) \dfrac{1}{(-4-1)} \sin 2x$

$= -\dfrac{1}{3}(D-1) \cdot \dfrac{1}{-5} \sin 2x$

$= \dfrac{1}{15}(D-1) \sin 2x$

$\therefore$ P.I. $= \dfrac{1}{15}(2\cos 2x - \sin 2x)$

Q4 $\dfrac{d^2y}{dx^2} + a^2 y = \cos ax$

$\therefore (D^2+a^2)y = \cos ax$

Eq. is of the form

$f(D)\, y = f(x)$

C.F. is obtained by

$f(D)\, y_c = \cancel{f(x)}\ 0$

$\therefore (D^2+a^2) y_c = 0$

A.E. is given by

$D^2+a^2=0 \quad \therefore D^2=-a^2$

$\therefore D=\pm ia$

$\therefore$ Roots are $m_1=\pm ia$

$\therefore$ C.F. is given by

$y_c=e^{0x}(c_1\cos ax+c_2\sin ax)$

$y_c=c_1\cos ax+c_2\sin ax.$

P.I. is given by

$f(D)\,y_p=f(x)$

$\therefore y_p=\dfrac{1}{D^2+a^2}\cos ax$

$\dfrac{1}{f(D)}\cos(ax+b)=\dfrac{1}{\phi(-a^2)}\dfrac{(-1)^r}{(2a)^r}\dfrac{x^r}{r!}\cos\left(ax+b+\dfrac{r\pi}{2}\right)$

$\therefore y_p=\dfrac{1}{1}\dfrac{(-1)^1}{(2a)^1}\cdot\dfrac{x}{1!}\cos\left(ax+\dfrac{\pi}{2}\right)$

$=\dfrac{(-1)}{2a}\cdot x(-1)\sin ax=\dfrac{x}{2a}\sin ax$

$\therefore$ G.S. is given by $y=y_c+y_p$

$\therefore y=c_1\cos ax+c_2\sin ax+\dfrac{x}{2a}\sin ax$

Q8. $\dfrac{d^2y}{dx^2}+y=\sin x.\sin 2x$

$(D^2+1)y=\dfrac{1}{2}(2\sin x\sin 2x)$

Eq. is of the form $f(D)y=f(x)$

C.F. ~~P.I.~~ is given by

$f(D)\,y_c=0$

A.E. is given by

$D^2+1=0 \quad \therefore D=\pm i$

$\therefore$ Roots are $m_1=\pm i$

$\therefore$ C.F. is $y_c=c_1\cos x+c_2\sin x.$

P.I. is given by

$f(D)\,y_p=f(x)$

$\therefore y_p=\dfrac{1}{D^2+1}\cdot\dfrac{1}{2}(\cos x-\cos 3x)$

$=\dfrac{1}{2}\left[\dfrac{1}{D^2+1}\cdot\cos x-\dfrac{1}{D^2+1}\cdot\cos 3x\right]$

$\dfrac{1}{f(D)}\cos(ax+b)=\dfrac{1}{f(-a^2)}\cos(ax+b)$

$$\frac{1}{f(D)}\cos(ax+b) = \frac{1}{\phi(-a^2)}\frac{(-1)^r}{(2a)^r}\cdot\frac{x^r}{r!}\cdot\cos\left(ax+b+\frac{r\pi}{2}\right)$$

$$\therefore y_p = \frac{1}{2}\left[\frac{(-1)^1}{2} \cdot \frac{x}{1!}\cos\left(x+\frac{\pi}{2}\right) - \frac{1}{-9+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{1}{8}\cos 3x\right]$$

$\therefore$ G.S. is given by : $y = y_c + y_p$

$$\therefore y = C_1\cos x + C_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

Q6 $(D^3+3D)y = \cosh 2x$.

Eq. is of the form

$$f(D)y = f(x)$$

C.F. is given by

$$f(D)y_c = 0$$

A.E. is $D^3+3D=0$

$$\therefore D(D^2+3)=0$$

$$\therefore D = 0, \pm i\sqrt{3}$$

$\therefore$ Roots are $m_1 = 0$, $m_2 = \pm i\sqrt{3}$

$\therefore$ C.F. is

$$y_c = C_1e^{0x} + e^{0x}(C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x)$$

$$= C_1 + C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x.$$

P.I. is given by

$$f(D)y_p = f(x)$$

$$\therefore y_p = \frac{1}{D^3+3D}\cosh 2x = \frac{1}{D^2\cdot D+3D}\cosh 2x$$

$$\therefore y_p = \frac{1}{+4D+3D}\cosh 2x = \frac{1}{7D}\cosh 2x$$

$$= \frac{1}{7}\int \cosh 2x\,dx = \frac{1}{7}\frac{\sinh 2x}{2} = \frac{\sinh 2x}{14}$$

$\therefore$ G.S. is $y = y_c + y_p$

$$\therefore y = C_1 + C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x + \frac{\sinh 2x}{14}$$

Q7. P.I. $(D^3-2D+4)y = 3x^2-5x+2$.

Eq. is of the form

$$f(D)y = f(x)$$

$$y_p = \frac{1}{4-2D+D^3}(3x^2-5x+2)$$

$$= \frac{1}{4\left(1-\frac{2D}{4}+\frac{D^3}{4}\right)}(3x^2-5x+2) = \frac{1}{4\left[1-\left(\frac{D}{2}-\frac{D^3}{4}\right)\right]}(3x^2-5x+2)$$

P.I. is given by

$$f(D)\,y_p = f(x)$$

$$\therefore y_p = \frac{1}{(D^3-2D+4)}(3x^2-5x+2)$$

$$\therefore y_p = \frac{1}{4}\left\{1-\left(\frac{D}{2}-\frac{D^3}{4}\right)\right\}^{-1}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1+\left(\frac{D}{2}-\frac{D^3}{4}\right)+\left(\frac{D}{2}-\frac{D^3}{4}\right)^2\dots\right\}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1+\frac{D}{2}+\frac{D^2}{4}\dots\right\}(3x^2-5x+2)$$

$$= \frac{1}{4}\left(3x^2-5x+2+3x-\frac{5}{2}+\frac{3}{2}\right)$$

$$\text{P.I.} = \frac{1}{4}(3x^2-2x+1)$$

• Q8. $(D^2-4)y = x^2e^{3x}$

Eq. is of form

$$f(D)\,y = f(x)$$

C.F. is given by

$$f(D)\,y_c = 0$$

A.E. is $D^2-4=0$

$$\therefore D = +2, -2$$

$\therefore$ Roots are $m_1 = 2$, $m_2 = -2$

$\therefore$ C.F. is $y_c = c_1e^{2x}+c_2e^{-2x}$.

P.I. is given by

$$f(D)\,y_p = f(x)$$

$$\therefore y_p = \frac{1}{(D^2-4)}\cdot e^{3x}x^2.$$

$$\frac{1}{f(D)}e^{ax}f(x) = e^{ax}\frac{1}{f(D+a)}\cdot f(x)$$

$$y_p = e^{3x}\frac{1}{(D+3)^2-4}x^2.$$

$$y_p = e^{3x}\frac{1}{(D^2+6D+5)}\cdot x^2$$

$$= e^{3x}\cdot\frac{1}{5}\left\{1+\left(\frac{6D}{5}+\frac{D^2}{5}\right)\right\}^{-1}\cdot x^2$$

$$= \frac{e^{3x}}{5}\left\{1-\left(\frac{6D}{5}+\frac{D^2}{5}\right)+\left(\frac{6D}{5}+\frac{D^2}{5}\right)^2\dots\right\}$$

$$y_p = \frac{e^{3x}}{5}\left\{1 - \frac{6D}{5} - \frac{D^2}{5} + \frac{36D^2}{25} \cdots\right\}x^2$$

$$= \frac{e^{3x}}{5}\left(1 - \frac{6D}{5} + \frac{31D^2}{25} \cdots\right)x^2$$

$$= \frac{e^{3x}}{5}\left(x^2 - \frac{12x}{5} + \frac{62}{25}\right)$$

G.S. is given by

$$y = y_c + y_p$$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{3x}}{5}\left(x^2 - \frac{12x}{5} + \frac{62}{25}\right)$$

(HW)

Q9. $(D^2+3D+2)y = x\sin 2x$.

Eq. is of the form

$$f(D)\, y = f(x)$$

C.F. is given by

$$f(D)\, y_c = 0$$

A.E. is $D^2+3D+2=0$

$$\therefore (D+2)(D+1)=0$$

$$\therefore D = -1, -2$$

$\therefore$ Roots are $m_1 = -1,\ m_2 = -2$

$\therefore$ C.F. is $y_c = c_1 e^{-x} + c_2 e^{-2x}$.

P.I. is given by

$$f(D)\, y_p = x\sin 2x$$

$$\therefore y_p = \frac{1}{(D^2+3D+2)} \cdot x\sin 2x$$

$$\frac{1}{f(D)} \cdot x f(x) = \left\{x - \frac{1}{f(D)} \cdot f'(D)\right\}\frac{1}{f(D)} f(x)$$

$$\therefore y_p = \left\{x - \frac{2D+3}{D^2+3D+2}\right\} \cdot \frac{1}{D^2+3D+2} \sin 2x$$

$$= x \cdot \frac{1}{D^2+3D+2}\sin 2x - \frac{2D+3}{(D^2+3D+2)^2} \cdot \sin 2x$$

$$= x \cdot \frac{1}{3D-2} \cdot \frac{3D+2}{3D+2}\sin 2x - \frac{2D+3}{(3D-2)^2}\sin 2x$$

$$= x \cdot \frac{3D+2}{9D^2-4}\sin 2x - \frac{2D+3}{9D^2-12D+4} \cdot \sin 2x$$

$$= x\, \frac{3D+2}{-40}\sin 2x - \frac{2D+3}{-32-12D} \cdot \sin 2x.$$

$$= \frac{x}{-40}(3D+2)\sin 2x + \frac{(2D+3)}{4(8D+3D)(3D-8)}(3D-8)\sin 2x$$

$$= \frac{x}{-40}(3.2\cos 2x + 2\sin 2x) + \frac{(6D^2-7D-24)}{4(9D^2-64)}\sin 2x$$

$$= \frac{x}{-40}(6\cos 2x + 2\sin 2x) + \frac{(-24-7D-24)}{4(9\times -4-64)}\sin 2x$$

$$= -\frac{x}{20}(3\cos 2x + \sin 2x) + \frac{(7D+48)}{4(100)}\sin 2x$$

$$= \frac{-x}{20}(3\cos 2x + \sin 2x) + \frac{1}{400}(7.2\cos 2x + 48\sin 2x)$$

$$= \frac{-x}{20}(3\cos 2x + \sin 2x) + \frac{1}{400}(14\cos 2x + 48\sin 2x)$$

$$= \cos 2x\left(\frac{14}{400} - \frac{3x}{20}\right) + \sin 2x\left(\frac{48}{400} - \frac{x}{20}\right)$$

$$= \cos 2x\left(\frac{7-30x}{200}\right) + \sin 2x\left(\frac{12-5x}{100}\right)$$

G.S. is $y = y_c + y_p$

$$\therefore y = c_1e^{-2x} + c_2e^{-x} + \left(\frac{7-30x}{200}\right)\cos 2x + \left(\frac{12-5x}{100}\right)\sin 2x$$

Eg. V-B.

Q1 $\frac{d^2y}{dx^2} + 4y = \sin 3x + e^x + x^2$

$(D^2+4)y = \sin 3x + e^x + x^2$

Eq. is of the form

$f(D)\,y = f(x)$

C.F. is given by $f(D)\,y_c = 0$

A.E. is $D^2+4=0$

$\therefore D = \pm 2i$

$\therefore y_c = c_1\cos 2x + c_2\sin 2x.$

P.I. is given by

$f(D)\,y_p = f(x)$

$$\therefore y_p = \frac{1}{(D^2+4)}(\sin 3x + e^x + x^2)$$

$$= \frac{1}{D^2+4}\sin 3x + \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}x^2.$$

$$= \frac{1}{-9+4}\sin 3x + \frac{1}{1^2+4}e^x + \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2$$

$$= \frac{-\sin 3x}{5} + \frac{e^x}{5} + \frac{1}{4}\left(1-\frac{D^2}{4}\ldots\right)x^2$$

$\therefore y_p = -\frac{\sin 3x}{5} + \frac{e^x}{5} + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)$

G.S. is : $y = y_c + y_p$

$\therefore y = c_1 \cos 2x + c_2 \sin 2x - \frac{\sin 3x}{5} + \frac{e^x}{5} + \frac{x^2}{4} - \frac{1}{8}$

• 2 $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$

$(D^4 + 2D^2 + 1)y = x^2 \cos x$

Eq. is of the form

$f(D) y = f(x)$

A.E. is $D^4 + 2D^2 + 1 = 0$

$\therefore (D^2+1)^2 = 0$

$\therefore D = \pm i, \pm i$

$\therefore$ Roots are $m_1 = \pm i$, repeated twice

$\therefore$ C.F. is $y_1 = (c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x$

P.I. is given by

$f(D)\, y_p = f(x)$

$\therefore y_p = \frac{1}{D^4 + 2D^2 + 1} x^2 \cos x.$

$= \cancel{\frac{1}{D^4+2D^2+1} \cdot x(x\cos x)}$

$= \cancel{x \left\{ x - \frac{4D^3 + 4D}{D^4+2D^2+1} \right\} \frac{1}{D^4+2D^2+1} x\cos x}$

$y_p = \frac{1}{(D^2+1)^2} x^2 \, \text{R.P.} \left[e^{ix}\right]$

$= \text{R.P.} \left[\frac{1}{(D^2+1)^2} e^{ix} \cdot x^2\right]$

$= \text{R.P.}\, e^{ix} \left[\frac{1}{((D+i)^2+1)^2} \cdot x^2\right]$

$= \text{R.P.}\, e^{ix} \left[\frac{1}{(D^2 + 2iD)^2} x^2\right]$

$= \text{R.P.}\, e^{ix} \cdot \frac{1}{D^2}\left[\frac{1}{(2i+D)^2} \cdot x^2\right]$

$= \text{R.P.}\, e^{ix} \cdot \frac{1}{D^2}\left[\frac{1}{(2i)^2}\left(1 + \frac{D}{2i} + \frac{D^2}{\ldots}\right)^{-1} x^2\right]$

$\text{R.P.}\, e^{ix} \frac{1}{D^2}\left[\frac{1}{1+4iD+D^2}\right] x^2$

$= \text{R.P.}\, \frac{e^{ix}}{4D^2}\left[\frac{1}{1-(iD+\frac{D^2}{4})}\right] x^2$

$= \text{R.P.}\, \frac{-1}{4} \frac{e^{ix}}{D^2}\left[1-(iD+\frac{D^2}{4})\right]^{-1} x^2$

$= \text{R.P.}\, \frac{-1}{4} \frac{e^{ix}}{D^2}\left[1 + iD + \frac{D^2}{4} - D^2\right] x^2$

$\left[x^2 + 2ix - \ldots\right]$

$\frac{x^4}{12} + i\frac{x^3}{3} - \ldots$

Expand $\left(2i + D\right)^{-2}$

$$-\frac{D^2}{4i^2}+\frac{D^2}{i^2}$$

$$y_p = R.P.\ e^{ix}\cdot\frac{1}{D^2}\left[\frac{1}{(2i)^2}\left(1-\frac{D}{2i}-\frac{D^2}{4i^2}+\frac{D^2}{4i^2}\right)x^2\right]$$

$$= R.P.\ \frac{e^{ix}}{(2i)^2}\cdot\frac{1}{D^2}\left(x^2-\frac{2x}{i}-\frac{3}{2}\right)$$

$$= R.P.\ \frac{e^{ix}}{(2i)^2}\frac{1}{D}\left(\frac{x^3}{3}-\frac{2x^2}{2i}-\frac{3x}{2}\right)$$

$$-\frac{3x^2}{4}$$

$$= R.P.\left[-\frac{1}{4}(\cos x+i\sin x)\left(\frac{x^4}{12}-\frac{2x^3}{6i}-\frac{3x^2}{4}\right)\right]$$

$$= R.P.\ -\frac{1}{4}\left(\cos x+i\sin x\right)\left(\frac{x^4}{12}-\frac{x^3}{3i}-\frac{3x^2}{4}\right)$$

$$= \cancel{-\frac{1}{4}\left[\left(\frac{x^3}{3}-\frac{x}{2}\right)\sin x+\frac{x^2}{2}\cos x\right]}$$

$$= -\frac{x^4}{48}\cos x+\frac{x^3\sin x}{12}+\frac{3x^2}{16}\cos x$$

$$y_p = \frac{x^3\sin x}{12}+\frac{9x^2-x^4}{48}\cos x$$

G.S. $y = y_c + y_p$

$$= (c_1x+c_2)\cos x+(c_3x+c_4)\sin x + \frac{x^3\sin x}{12}+\frac{9x^2-x^4}{48}\cos x$$

3. $$\frac{d^3y}{dx^3}-7\frac{dy}{dx}-6y = e^{2x}(1+x)$$

$$(D^3-7D-6)y = e^{2x}(1+x)$$

Eq. is of the form

$$f(D)y = f(x)$$

	1	0	-7	-6
	0	-1	1	6
-1	1	-1	-6	0

CF is given by

$$f(D)\,y_c = 0$$

A.E. is $D^3-7D-6=0$

$\cancel{D^2-7D-6=0}$

$\therefore \cancel{(D-6)(D+1)\ D=\frac{7\pm\sqrt{49+24}}{2}}$

$\cancel{D=\frac{7\pm\sqrt{73}}{2}}$ $(D+1)(D^2-D-6)=0$

$$D=-1,\ \frac{1\pm\sqrt{1+24}}{2} = -1,\ \frac{1\pm5}{2}$$

$$D=-1,3,-2$$

$$y_c = c_1e^{-x}+c_2e^{-2x}+c_3e^{3x}$$

P.I. is given by

$f(D)\, y_P = f(X)$

$\therefore y_P = \dfrac{1}{(D+1)(D+2)(D-3)} \cdot e^{2x}(1+x)$

$= \dfrac{1}{(D+1)(D+2)(D-3)} \cdot e^{2x} + \dfrac{1}{(D+1)(D+2)(D-3)} \cdot e^{2x} \cdot x$

$= \dfrac{1}{3\times4\times(-1)} e^{2x} + e^{2x} \cdot \dfrac{1}{(D+3)(D+4)(D-1)} \cdot x$

$= \dfrac{-e^{2x}}{12} + e^{2x} \dfrac{1}{(D^2+7D+12)(D-1)} \cdot x$

$= -\dfrac{e^{2x}}{12} + e^{2x} \dfrac{1}{(D^3+7D^2+12D-D^2-7D-12)} \cdot x$

$= -\dfrac{e^{2x}}{12} + e^{2x} \dfrac{1}{(D^3+6D^2+5D-12)} \cdot x$

$= -\dfrac{e^{2x}}{12} + e^{2x}\left(\dfrac{-1}{12}\right)\left[1-\left(\dfrac{5D}{12}+\dfrac{D^2}{2}+\dfrac{D^3}{12}\right)\right]^{-1} x$

$= \dfrac{-e^{2x}}{12} - \dfrac{e^{2x}}{12}\left[1+\dfrac{5D}{12}\right] \cdot x$

$= \dfrac{-e^{2x}}{12} - \dfrac{e^{2x}}{12}\left(x+\dfrac{5}{12}\right)$

$= \dfrac{-e^{2x}}{12}\left(x+\dfrac{17}{12}\right)$

$\therefore$ G.S. is $y = y_c + y_p$

$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \dfrac{e^{2x}}{12}\left(x+\dfrac{17}{12}\right)$

4 $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

$(D^3+1)\, y = 3 + e^{-x} + 5e^{2x}$

Eq. is of the form $f(D)\, y = f(X)$

C.F. is given by

$f(D)\, y_c = 0$

A.E. is $D^3+1 = 0$

$(D+1)(D^2-D+1) = 0$

$\therefore D = -1,\ \dfrac{1 \pm \sqrt{1-4}}{2}$

$D = -1,\ \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$\therefore y_c = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right)$

P.I. is given by

$f(D)\, y_p = f(x)$

$\therefore y_p = \frac{1}{(D^3+1)}(3 + e^{-x} + 5e^{2x})$

$\therefore y_p = \frac{1}{(D^3+1)} 3e^{0x} + \frac{1}{(D^3+1)} e^{-x} + 5 \cdot \frac{1}{(D^3+1)} e^{2x}$

$= 3 \frac{1}{0+1} \cdot e^{0x} + \frac{1}{(D+1)(D^2-D+1)} \cdot e^{-x} + 5 \frac{1}{9} e^{2x}$

$= 3 + \frac{1}{3} \cdot \frac{x^1}{1!} e^{-x} + \frac{5}{9} e^{2x}$

$\therefore y_p = 3 + \frac{x e^{-x}}{3} + \frac{5}{9} e^{2x}$

G.S. is $y = y_c + y_p$

$\therefore y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right)$

$+ 3 + \frac{x e^{-x}}{3} + \frac{5e^{2x}}{9}$

• 5 $\frac{d^2y}{dx^2} + 4y = x \sin x$.

$(D^2+4)\, y = x \sin x$

Eq is of the form $f(D)\, y = f(x)$

C.F. is given by $f(D)\, y_c = 0$

A.E. is $D^2 + 4 = 0$

$\therefore D = \pm 2i$

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$

P.I. is given by $f(D)\, y_p = f(x)$

$\therefore y_p = \frac{1}{D^2+4} \cdot x \sin x$

$= \frac{1}{D^2+4} \cdot x \; \text{I.P.}\left[e^{ix}\right]$

$= \text{I.P.}\left[\frac{1}{D^2+4} \cdot e^{ix} \cdot x\right]$

$= \text{I.P.}\left[e^{ix} \frac{1}{(D+i)^2+4} \cdot x\right]$

$= \text{I.P.}\left[e^{ix} \frac{1}{D^2+2iD+3} \cdot x\right]$

$$y_p = I.P.\left[e^{ix}\frac{1}{D}\cdot\frac{1}{(D+4i)}x\right]$$

$$= I.P.\left[e^{ix}\frac{1}{D}\cdot\frac{1}{4i}\left(1+\frac{D}{4i}\right)^{-1}x\right]$$

$$= I.P.\left[\frac{e^{ix}}{4i}\cdot\frac{1}{D}\left(1-\frac{D}{4i}\right)x\right]$$

$$= I.P.\left[\frac{e^{ix}}{4i}\cdot\frac{1}{D}\left(x-\frac{1}{4i}\right)\right]$$

$$= I.P.\left[\frac{e^{ix}}{4i}\left(\frac{x^2}{2}-\frac{x}{4i}\right)\right]$$

$$= \frac{1}{4}I.P.\left[\left(\frac{\cos x + i\sin x}{i}\right)\left(\frac{x^2}{2}-\frac{x}{4i}\right)\right]$$

$$= -\frac{1}{4}I.P.\left[(i\cos x - \sin x)\left(\frac{x^2}{2}-\frac{x}{4i}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{x^2}{2}\cos x - \frac{x\sin x}{4}\right]$$

$$y_p = I.P.\left[e^{ix}\frac{1}{3+2iD+D^2}\cdot x\right]$$

$$= I.P.\left[e^{ix}\frac{1}{3}\frac{1}{\left(1+\frac{2iD}{3}+\frac{D^2}{3}\right)}\cdot x\right]$$

$$= I.P.\left\{\frac{e^{ix}}{3}\left[1+\left(\frac{2iD}{3}+\frac{D^2}{3}\right)\right]^{-1}\cdot x\right\}$$

$$= I.P.\left\{\frac{e^{ix}}{3}\left[1-\frac{2iD}{3}\right]\cdot x\right\}$$

$$= I.P.\left[\frac{e^{ix}}{3}\left(x-\frac{2i}{3}\right)\right]$$

$$= \frac{1}{3}I.P.\left[(\cos x + i\sin x)\left(x-\frac{2i}{3}\right)\right]$$

$$= \frac{1}{3}\left[-\frac{2}{3}\cos x + x\sin x\right]$$

$$\therefore y_p = \frac{x\sin x}{3} - \frac{2\cos x}{9}.$$

G.S. $y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{x\sin x}{3} - \frac{2\cos x}{9}$

6 $\frac{d^2y}{dx^2} - y = x^2\sin 3x \quad \therefore (D^2-1)y = x^2\sin 3x$

Eq. is of the form $f(D)y = f(X)$

C.F. is given by $f(D)\,y_c = 0$

A.E. is $D^2 - 1 = 0 \quad \therefore D = 1, -1$

$\therefore$ Roots are $m_1 = 1$, $m_2 = -1$

$\therefore y_c = c_1 e^{x} + c_2 e^{-x}$

P.I. is given by $f(D)y_p = f(X)$

$\therefore y_p = \frac{1}{(D^2-1)} x^2\sin 3x$

$= \frac{1}{(D^2-1)} \cdot x^2 \, I.P. \left[e^{i3x}\right]$

$= I.P. \left\{ \frac{1}{(D^2-1)} e^{i3x} \cdot x^2 \right\}$

$= I.P. \left\{ \frac{1}{(D+1)(D-1)} \cdot e^{3ix} \cdot x^2 \right\}$

$= I.P. \left\{ e^{3ix} \frac{1}{(D+3i+1)(D+3i-1)} \cdot x^2 \right\}$

$\therefore y_p = I.P. \left\{ e^{3ix} \frac{1}{[(D+3i)^2 - 1]} \cdot x^2 \right\}$

$= I.P. \left\{ e^{3ix} \frac{1}{D^2 + 6iD - 10} \cdot x^2 \right\}$

$= I.P. \left\{ e^{3ix} \frac{1}{-10(1 - \frac{6iD}{10} - \frac{D^2}{10})} \cdot x^2 \right\}$

$= -\frac{1}{10} \, I.P. \left\{ e^{3ix} \left[1 - \left(\frac{6iD}{10} + \frac{D^2}{10}\right)\right]^{-1} x^2 \right\}$

$= -\frac{1}{10} \, I.P. \left\{ e^{3ix} \left[1 + \frac{6iD}{10} + \frac{D^2}{10} + \frac{36i^2D^2}{100}\right] x^2 \right\}$

$= -\frac{1}{10} \, I.P. \left\{ e^{3ix} \left[1 + \frac{6iD}{10} - \frac{26}{100}D^2\right] x^2 \right\}$

$= -\frac{1}{10} \, I.P. \left\{ e^{3ix} \left(x^2 + \frac{12ix}{10} - \frac{\cancel{50}\,52}{100}\right) \right\}$

$= -\frac{1}{10} \, I.P. \left\{ (\cos 3x + i\sin 3x)\left(x^2 + \frac{12}{10}ix - \frac{\cancel{50}\,52}{100}\right) \right\}$

$= -\frac{1}{10} \left(\frac{12}{10} x\cos 3x + x^2 \sin 3x - \frac{\cancel{50}\,52}{100} \sin 3x \right)$

$\therefore y_p = -\frac{\cancel{12}^{3}}{\cancel{100}_{25}} x\cos 3x - \frac{x^2}{10}\sin 3x + \frac{\cancel{52}^{13}}{10\times\cancel{100}_{25}}\sin 3x$

$= -\frac{3x\cos 3x}{25} - \frac{25x^2-13}{250}\sin 3x$

G.S. is $y = y_c + y_p$

$\therefore y = C_1e^{x} + C_2e^{-x} - \frac{3x\cos 3x}{25} - \frac{25x^2-13\sin 3x}{250}$

I $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Eq. is of the form $f(D)y = f(X)$

$(D^2+3D+2)y = e^{e^x}$

C.F is given by $f(D)y_c = 0$

A.E. is $D^2+3D+2=0$

$\therefore (D+2)(D+1) = 0$

$\therefore D = -2, -1$

$\therefore y_c = C_1e^{-x} + C_2e^{-2x}$

P.I. is given by $f(D)y_p = f(X)$

$\therefore y_p = \frac{1}{(D+2)(D+1)} \cdot e^{e^x}$

$= \left[\frac{-1}{D+2} + \frac{1}{D+1}\right] e^{e^x}$

$= \frac{1}{D+1} \cdot e^{e^x} - \frac{1}{D+2} e^{e^x}$

$\frac{1}{D+m} f(x) = e^{-mx}\int e^{mx} f(x)\,dx$

$= e^{-x}\int e^{x} e^{e^x}\,dx - e^{-2x}\int e^{2x} e^{e^x}\,dx$

Let $e^x = t$ $\therefore e^x dx = dt$

$y_p = e^{-x}\int e^t\,dt - e^{-2x}\int t\,e^t\,dt$

$= e^{-x}(e^t) - e^{-2x}(te^t - e^t)$

$= e^{-x} e^{e^x} - e^{-2x} e^x e^{e^x} + e^{-2x} e^{e^x}$

$= \cancel{e^{-x} e^{e^x}} - \cancel{e^{-x} e^{e^x}} + e^{-2x} e^{e^x}$

$y_p = e^{-2x} e^{e^x}$.

G.S. is $y = y_c + y_p$

$\therefore y = C_1e^{-x} + C_2e^{-2x} + e^{-2x} e^{e^x}$.

TYPE IV : Method of Variation of Parameters.

C.F is obtained by $f(D)y_c=0$

P.I. is obtained by subs. the arbitary const. of C.F. by u, v, w etc. which are functions of x. y_p is diff. & conditional eqs. are formed in order to get u, v, w etc.

33 Eg1 $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

Eq. is of the form $f(D)y = f(X)$

$(D^2+a^2)y = \sec ax$

C.F is given by $f(D)y_c=0$

A.E. is $D^2+a^2=0$

$\therefore D = \pm ia$

$\therefore y_c = C_1\cos ax + C_2 \sin ax$

By method of variation of parameters:-

$y_p = u\cos ax + v\sin ax$ —(I)

where u, v are functions of x.

Diff (I) wrt x

$\therefore Dy_p = -ua\sin ax + va\cos ax + (u'\cos ax + v'\sin ax)$

Consider

$u'\cos ax + v'\sin ax = 0$ —(II)

{NOTE: Consider u' & v' term $=0$ in order to avoid its second order der on diff. again}.

$\therefore Dy_p = -ua\sin ax + va\cos ax$ —(III)

Diff. (III) wrt x {We diff. twice as D^2 terms are in quest}

$D^2y_p = -ua^2\cos ax - va^2\sin ax - u'a\sin ax + v'a\cos ax$ —(IV)

From (I) & (IV).

$a^2y_p + D^2y_p = -u'a\sin ax + v'a\cos ax$

Also, $D^2 y_p + a^2 y_p = \sec ax$

$\cos ax \times (-u' a \sin ax + v' a \cos ax = \sec ax)$

$a \sin ax \times (u' \cos ax + v' \sin ax = 0)$

$\therefore v' a = 1 \quad \therefore v' = 1/a$

$u' = -\dfrac{\sin ax}{a \cos ax} = -\dfrac{1}{a} \tan ax$

$\therefore u = \int u' dx = -\dfrac{1}{a} \int \tan ax = \dfrac{-1}{a^2} \log(\sec ax)$

$v = \int v' dx = \dfrac{1}{a} \int dx = \dfrac{x}{a}$

From (I)

$y_p = \dfrac{-\cos ax}{a^2} \log(\sec ax) + \dfrac{x \sin ax}{a}$

$\therefore$ G.S. is $y = y_c + y_p$

$\therefore y = c_1 \cos ax + c_2 \sin ax - \dfrac{\cos ax}{a^2} \log(\sec ax) + \dfrac{x \sin ax}{a}$

Ex. II - C

1 $(D^2+4) y = 4 \sec^2 2x$

Eq is of the form

$f(D) y = f(x)$

C.F. is given by

$f(D) y_c = 0$

A.E. is $D^2+4 = 0$

$\therefore D = \pm 2i$

$\therefore$ C.F. is $y_c = c_1 \cos 2x + c_2 \sin 2x$

~~P.I. is given by~~

~~$f(D) y_p = f(x)$~~

By method of var. of param.

$y_p = u \cos 2x + v \sin 2x$ — (I)

where u, v are functs. of x

Diff (I) wrt x.

$\therefore D y_p = -2u \sin 2x + 2v \cos 2x + (u' \cos 2x + v' \sin 2x)$

Consider $u' \cos 2x + v' \sin 2x = 0$ — (II)

$\therefore Dy_p = -2u\sin 2x + 2v\cos 2x$ — (III)

Diff. (III) w.r.t x

$\therefore D^2 y_p = $ [illegible] $\cos 2x - 4v\sin 2x$

[illegible] $\sin 2x + 2v'\cos 2x$ — (IV)

From [illegible] (IV)

[illegible] $+ 4y_p = -$[illegible]$u'\sin 2x + 2v'\cos 2x$

Also, $(D^2$ [illegible]$)y_p = 4\sec^2 2x$

$\cos 2x$ [illegible] $-u'\sin 2x + v'\cos 2x = 2\sec^2 2x$)

$\sin$ [illegible] $2x + v'\sin 2x = 0$)

[illegible] $= 2\sec 2x$.

[illegible] $u' = -2$[illegible]

[illegible] $2x$

$\therefore v = \int v'\,dx = 2\int \sec 2x\,dx = \frac{2}{2}\log(\sec 2x + \tan 2x)$

$u = \int u'\,dx = -2\int \frac{\sin 2x}{\cos^2 2x}\,dx = -\frac{1}{\cos 2x}$

From (I), P.I. is $y_p = -1 + \sin 2x \log(\sec 2x + \tan 2x)$

G.S. is $y = y_c + y_p$

$\therefore y = c_1\cos 2x$ [illegible] $\sin 2x - 1 + \sin 2x$ [illegible] $\log(\sec 2x + \tan 2x)$

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2 $(D^2 - 1)y = ($[illegible]

Eq is [illegible] the form

$f(D)\,y = f$[illegible]

C.F. is given by

$f(D)\,y_c = 0$

A.E. is D^2 [illegible]

$\therefore$ Roots are $m_1 =$ [illegible] -1

$\therefore y_c = c$[illegible]$ + c_2 e^{-x}$

By method [illegible] var. of [illegible]

$y_p = u e^{x} + v e^{-x}$ — (I)

where u, v are functions of x

Diff. (I) w.r.t [illegible]

$\therefore Dy_p = u e^{x} - v e^{-x} + (u' e^{x} + v' e^{-x})$

Consider, $u' e^{x} + v' e^{-x} = 0$ — (II)

$\therefore Dy_p = ue^{x} - ve^{-x}$ —(III)

Diff (III) wrt x

$\therefore D^2y_p = ue^{x} + ve^{-x} + u'e^{x} - v'e^{-x}$ —(IV)

From (I) & (II)

$D^2y_p - y_p = u'e^{x} - v'e^{-x}$

Also, $(D^2-1)\,y_p = (1+e^{-x})^{-2}$

$$\therefore \begin{aligned} u'e^{x} - v'\cancel{e^{-x}} &= (1+e^{-x})^{-2} \\ u'e^{x} + \cancel{v'}e^{-x} &= 0 \end{aligned}$$

$$u' = \frac{1}{2e^{x}}\cdot\frac{1}{(1+e^{-x})^2} = \frac{1}{2\cancel{e^{x}}}\,\frac{e^{\cancel{2}x}}{(e^{x}+1)^2}$$

$$v' = -\frac{e^{3x}}{2(e^{x}+1)^2}$$

$$u = \int u'\,dx = \frac{1}{2}\int\frac{e^{x}}{(e^{x}+1)^2}\,dx = -\frac{1}{2(e^{x}+1)}$$

$$v = \int v'\,dx = -\frac{1}{2}\int\frac{e^{3x}\,dx}{(e^{x}+1)^2} = -\frac{1}{2}\int\frac{e^{2x}e^{x}dx}{(e^{x}+1)^2}$$

Let $e^{x}+1 = t \quad \therefore e^{x}dx = dt$

$\therefore e^{2x} = (t-1)^2$

$$\therefore v = -\frac{1}{2}\int\frac{(t-1)^2}{t^2}\,dt = -\frac{1}{2}\int\frac{t^2-2t+1}{t^2}\,dt$$

$$= -\frac{1}{2}\left[t - 2\log t - \frac{1}{t}\right]$$

$$\therefore v = -\frac{1}{2}(e^{x}+1) + \log(e^{x}+1) + \frac{1}{2(e^{x}+1)}$$

From (I)

$$y_p = -\frac{e^{x}}{2(e^{x}+1)} - \frac{e^{-x}}{2}(e^{x}+1) + e^{-x}\log(e^{x}+1) + \frac{e^{-x}}{2(e^{x}+1)}$$

$$= e^{-x}\log(1+e^{x}) + \frac{e^{-x}}{2(e^{x}+1)} - \frac{e^{x}}{2(e^{x}+1)} - \frac{e^{-x}(e^{x}+1)}{2}$$

$$= e^{-x}\log(1+e^{x}) + \frac{1}{2(e^{x}+1)}\left(\frac{1}{e^{x}} - e^{x}\right) - \frac{e^{x}+1}{2e^{x}}$$

$$= e^{-x}\log(1+e^{x}) + \frac{1}{2(e^{x}+1)}\left(\frac{e^{2x}+1}{e^{x}}\right) - \frac{e^{x}+1}{2e^{x}}$$

$= e^{-x}\log(e^x+1) + \frac{1}{2e^x}\left(\frac{e^{2x}+1}{e^x+1} - e^x+1\right)$

$= e^{-x}\log(1+e^x) + \frac{1}{2e^x}\left(\frac{\cancel{e^{2x}}+1-\cancel{e^{2x}}-2e^x-1}{e^x+1}\right)$

$\cancel{= e^{-x}\log(1+e^x) + \frac{1}{2e^x}\left(\frac{-2e^x}{e^x+1}\right)}$

$= e^{-x}\log(1+e^x) + \frac{1}{2e^x}\left(\frac{-2e^{2x}-2e^x}{e^x+1}\right)$

$= e^{-x}\log(1+e^x) + \frac{1}{\cancel{2e^x}}(-\cancel{2e^x})\left(\frac{\cancel{e^x+1}}{\cancel{e^x+1}}\right)$

$y_p = e^{-x}\log(1+e^x) - 1$

G.S. is $y = y_c + y_p$

$\therefore y = c_1e^x + c_2e^{-x} - 1 + e^{-x}\log(1+e^x)$

36 3 $(D^3+D)y = \operatorname{cosec} x$

Eq. is of the form $f(D)y = f(X)$

C.F. is given by $f(D)\, y_c = 0$

A.E. is $D^3 + D = 0$

$D(D^2+1) = 0$

$D = 0, \pm i$

$\therefore y_c = c_1 + c_2\cos x + c_3 \sin x$

By method of var. of param.

$y_p = u + v\cos x + w\sin x$ — (I)

where u, v, w are functs. of x

Diff. (I) wrt x

$\therefore Dy_p = -v\sin x + w\cos x + (u' + v'\cos x + w'\sin x)$ — (II)

Consider $u' + v'\cos x + w'\sin x = 0$

$\therefore Dy_p = -v\sin x + w\cos x$ — (III)

Diff (III) wrt x

$D^2y_p = -v\cos x - w\sin x + (v'\sin x + w'\cos x)$ — (IV)

Consider,

$\therefore -v'\sin x + w'\cos x = 0$

Diff (IV) wrt x

$\therefore D^3 y_p = v\sin x - w\cos x$
$\quad - v'\cos x - w'\sin x$

$D^3 y_p + D y_p = -v'\cos x - w'\sin x$

$\sin x \times \left(\therefore -v'\cos x - w'\sin x = \operatorname{cosec} x\right)$
$\cos x \times \left(-v'\sin x + w'\cos x = 0\right)$

$\therefore -w' = 1 \quad \therefore w = -x$

$v' = -\dfrac{\cos x}{\sin x}$

$v = \int v'\,dx = -\int \cot x\,dx = -\log(\sin x)$

$\therefore u' + v'\cos x + w'\sin x = 0$

$\therefore u' = \sin x + \dfrac{\cos^2 x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x}$

$\therefore u' = \operatorname{cosec} x$ (2)

$u = \int u'\,dx = \int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x + \cot x)$

$\therefore y_p = \log(\operatorname{cosec} x + \cot x) - \cos x \log \sin x$
$\quad - x\sin x$

G.S. is $y = y_c + y_p$

$\therefore y = c_1 + c_2\cos x + c_3\sin x + \log(\operatorname{cosec} x + \cot x)$
$\quad - \cos x \log(\sin x) - x\sin x$.

37 Q $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Eq. is of the form $f(D)y = f(X)$

C.F. is given by
$f(D)y_c = 0$

$\therefore$ A.E. is $D^2 - 1 = 0 \quad \therefore D = \pm 1$

$\therefore y_c = c_1 e^{-x} + c_2 e^{x}$

By method of var. of param.

$y_p = ue^{-x} + ve^{x}$ — (I)

where u, v are functions of x.

Diff (I) wrt x

$Dy_p = -ue^{-x} + ve^{x} + (u'e^{-x} + v'e^{x})$

Consider

$u'e^{-x} + v'e^{x} = 0 \quad \text{—(II)}$

$\therefore Dy_p = -ue^{-x} + ve^{x} \quad \text{—(III)}$

Diff (III) wrt x

$\therefore D^2y_p = ue^{-x} + ve^{x} - u'e^{-x} + v'e^{x} \quad \text{—(IV)}$

From (I) & (IV)

$D^2y_p - y_p = -u'e^{-x} + v'e^{x}$

$\therefore -u'e^{-x} + v'e^{x} = e^{-x}\sin e^{-x} + \cos e^{-x}$

$\cancel{u'e^{-x}} + v'e^{x} = 0$

$\therefore v' = \frac{1}{2e^{x}}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)$

$v = \int v'\,dx = \frac{1}{2}\int \left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)e^{-x}\,dx$

Let $e^{-x} = t \quad \therefore -e^{-x}dx = dt$

$\therefore v = \frac{-1}{2}\int (t\sin t + \cos t)\,dt$

$= \frac{-1}{2}\left[-t\cos t + \sin t + \sin t\right]$

$\therefore v = -\frac{1}{2}\left[-e^{-x}\cos e^{-x} + 2\sin e^{-x}\right]$

$u' = -e^{2x}\frac{1}{2e^{x}}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)$

$u = \int u'\,dx = -\frac{1}{2}\int e^{x}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)dx$

Let $e^{-x} = t \quad \therefore -e^{-x}dx = dt$

$\therefore -t\,dx = dt \quad \therefore dx = -\frac{dt}{t}$

$\therefore e^{x} = 1/t$

$u = \frac{-1}{2}\int \sin e^{-x}\,dx - \frac{1}{2}\int e^{x}\cos e^{-x}\,dx$

$= \frac{-1}{2}\int \sin t\left(\frac{-dt}{t}\right) - \frac{1}{2}\int \frac{1}{t}\cos t\left(-\frac{dt}{t}\right)$

$= \frac{1}{2}\int \frac{1}{t_u}\sin t_v\,dt + \frac{1}{2}\int \frac{1}{t^2}\cos t\,dt$

$= \frac{1}{2}\left[\frac{1}{t}(-\cos t) - \cancel{\int + \frac{1}{t^2}(+\cos t)\,dt}\right] + \frac{1}{2}\int \frac{1}{t^2}\ldots$

$\therefore u = -\frac{1}{2e^{-x}} \cos e^{-x}$

$\therefore y_p = -\frac{1}{\cancel{2e^{-x}}} \cancel{e^{-x}} \cancel{\cos e^{-x}} + \cancel{\frac{1}{2} \cos e^{-x}}$

$- e^{x} \sin e^{-x}$.

GS is $y = y_c + y_p$

$\therefore y = c_1 e^{x} + c_2 e^{-x} - e^{x} \sin e^{-x}$.

~~TYPE~~ V : METHOD OF UNDETERMINED COEFFICIENTS (MULTIPLIERS)

[WARTIKAR—II Pg. 96, 97].

P.T.O.

TYPE VI :- Cauchy's linear Eq. (Homogeneous Eq.) -

Eq. of the type -

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = f(x)$$

where $P_0, P_1, \dots P_n$ are constants is an homogeneous eq. which can be reduced to a linear D.E. with const. coeff. by subs.

$x = e^z$

$\therefore z = \log x$

$\therefore x\frac{dy}{dx} = Dy \; ; \; x^2\frac{d^2y}{dx^2} = D(D-1)$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)$

$\therefore x^r\frac{d^ry}{dx^r} = D(D-1)(D-2)\dots(D-r+1)$

where $D = \frac{d}{dz}$

Eg. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$

Eq. is an homogeneous eq.

Let $x = e^z \therefore z = \log x$.

$D = \frac{d}{dz}$

$\therefore [D(D-1) - 2D - 4]y = e^{2z} + 2z$

$\therefore (D^2 - 3D - 4)y = e^{2z} + 2z$.

Is a linear D.E. with C.C.

A.E. is $D^2 - 3D - 4 = 0$

$(D-4)(D+1) = 0 \quad \therefore D = 4 \; ; \; D = -1$.

$\therefore y_c = c_1 e^{-z} + c_2 e^{4z}$

$y_p = \frac{1}{(D^2-3D-4)} \cdot e^{2z} + 2 \cdot \frac{1}{D^2-3D-4} z$.

$= \frac{1}{4-6-4} e^{2z} - \frac{2}{4} \cdot \frac{1}{1+\left(\frac{3D}{4} - \frac{D^2}{4}\right)} \cdot z$

$= -\frac{e^{2z}}{6} - \frac{1}{2}\left[1 + \left(\frac{3D}{4} - \frac{D^2}{4}\right)\right]^{-1} \cdot z$

$y_p = -\frac{e^{2z}}{6} - \frac{1}{2}\left[1-\left(\frac{3D}{4}-\frac{D^2}{4}\right)\right]2.$

$y_p = -\frac{e^{2z}}{6} - \frac{1}{2}\left[2-\frac{3}{4}\right] = -\frac{e^{2z}}{6} - \frac{2}{2}+\frac{3}{8}$

G.S. is $y = y_c + y_p$

$\therefore y = c_1 e^{-z} + c_2 e^{4z} - \frac{e^{2z}}{6} - \frac{2}{2} + \frac{3}{8}$

$\because z = \log x$

$\therefore y = c_1 x^{-1} + c_2 x^4 - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}$

Eg. IV-D

$(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$

Eq. is a homogeneous eq.

Let $x = e^z$ $\therefore z = \log x$

$D = d/dz$

$\therefore [D(D-1)(D-2) + D(D-1) - 2]y = x + \frac{1}{x^3}$

$[D^3 - 3D^2 + 2D + D^2 - D - 2]y = x + 1/x^3.$

$[D^3 - 2D^2 + D - 2]y = x + 1/x^3.$

Is a L.D.E. with C.C.

A.E. is $D^3 - 2D^2 + D - 2 = 0$

$D^2(D-2) + 1(D-2) = 0$

$(D^2+1)(D-2) = 0$

$\therefore D = 2, \pm i$

$y_c = c_1 e^{2z} + (c_2 \cos z + c_3 \sin z)$

$\cancel{y_p = \frac{1}{2\left[1-\left(\frac{D}{2}+\frac{D^2}{2}-\frac{D^3}{2}\right)\right]}\left(x+\frac{1}{x^3}\right)(e^z+e^{-3z})}$

$y_p = \frac{1}{(D^2+1)(D-2)}\left[e^z + e^{-3z}\right]$

$= \frac{1}{(2)(-1)}e^z + \frac{1}{(10)(-5)}e^{-3z}.$

$= \frac{e^z}{-2} - \frac{e^{-3z}}{50}$

$y = y_c + y_p = c_1 x^2 + (c_2 \cos \log x + c_3 \sin \log x) - \frac{x}{2} - \frac{1}{50x^3}$

2. $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=3x-7$

Is a homogeneous eq.

Let $x=e^z$ $\therefore z=\log x$; $D=d/dz$

$[D(D-1)(D-2)+3D(D-1)+D-1]y=3e^z-7$

$[D^3-3D^2+2D+3D^2-3D+D-1]y=3e^z-7$

$[D^3-2D-1]y=3e^z-7$

A.E. is $D^3-2D-1=0$

1	0	-2	-1	-1
	0	+1	1	
1	-1	-1	0	

$(D+1)(D^2-D-1)=0$

$D=-1,\ \frac{1\pm\sqrt{1+4}}{2}=-1,\ \frac{1\pm\sqrt{5}}{2}$

$(D^3-1)y=3e^z-7$

$(D-1)(D^2+D+1)y=3e^z-7$

$D=1;\ D=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}$

$\therefore y_c=c_1e^z+e^{-1/2z}\left[c_1\cos\frac{\sqrt{3}}{2}z+c_2\sin\frac{\sqrt{3}}{2}z\right]$

$y_p=\frac{1}{(D-1)(D^2+D+1)}.3e^z-\frac{1}{D^3-1}.7e^{0z}$

$=3\frac{1}{3}\frac{z^1}{1!}.e^z-\frac{1}{(-1)}.7$

$=ze^z+7=x\log x+7$

C.S. is $y=y_c+y_p$

$\therefore y=c_1e^z+e^{-z/2}\left[c_1\cos\frac{\sqrt{3}}{2}z+c_2\sin\frac{\sqrt{3}}{2}z\right]$

$+x\log x+7$

$=c_1x+x^{-1/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\log x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$

$+x\log x+7.$

3. $(x^2D^2-xD+4)y=\cos(\log x)+x\sin(\log x)$

Is an homogeneous eq.

Let $x=e^z$, $z=\log x$, $D=d/dz$

$[D(D-1)-D+4]y=\cos z+e^z\sin z$

$[D^2-2D+4]y=\cos z+e^z\sin z$

A.E. is $D^2-2D+4=0$

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$$D = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$\therefore y_c = e^{z}(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z)$$

$$y_p = \frac{1}{D^2-2D+4}(\cos z + e^2 \sin z)$$

$$= \frac{1}{D^2-2D+4} \cdot \cos z + e^2 \frac{1}{(D+1)^2-2(D+1)+4} \cdot \sin z$$

$$= \frac{1}{3-2D} \cos z + e^2 \frac{1}{D^2+3} \sin z$$

$$= \frac{3+2D}{9-4D^2} \cos z + e^2 \frac{1}{2} \sin z$$

$$= \frac{(3+2D)}{13} \cos z + \frac{e^2}{2} \sin z$$

$$= \frac{3\cos z - 2\sin z}{13} + \frac{e^2}{2} \sin z.$$

$$= \frac{3\cos(\log x) - 2\sin(\log x)}{13} + \frac{x}{2}\sin(\log x)$$

G.S. is $y = y_c + y_p$

$$\therefore y = x(c_1 \cos\sqrt{3}\log x + c_2 \sin\sqrt{3}\log x) + \frac{3\cos(\log x) - 2\sin(\log x)}{13} - \frac{x}{2}\sin(\log x)$$

TYPE VIII: LEGENDRE Linear Eq:-

Eq. of the form -

$$P_0(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + P_n y = f(x)$$

where $P_0, P_1, \dots, P_n$ are const. is known as Legendre L.E.

Let $a+bx = t$, to reduce to homog. eq.

Let $e^z = t$ to red. to C.D.E. with C.C.

Eg 13 $(x+a)^2\frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x$

Let $x+a = t$ $\therefore dt/dx = 1$

$$t^2\frac{d^2y}{dt^2} - 4t\frac{dy}{dt} + 6y = t - a$$

Is an homogeneous eq.

Let $t = e^z$ $\therefore z = \log t$; $D = d/dz$

$\therefore [D(D-1) - 4D + 6]y = e^z - a$

$[D^2 - 5D + 6]y = e^z - a$

$(D-3)(D-2)y = e^z - a$

A.E. is $(D-3)(D-2) = 0$ $\therefore D = 2, 3$

$y_c = c_1 e^{2z} + c_2 e^{3z}$

$$y_p = \frac{1}{(D-3)(D-2)}(e^z - a)$$

$$= \frac{e^z}{2} - \frac{ae^0}{6}$$

G.S. is $y = y_c + y_p$

$$y = c_1 t^2 + c_2 t^3 + \frac{t}{2} - \frac{a}{6}$$

$$\therefore y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{3x+2a}{6}$$

Eg. VI - D.

4 $$(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$$

Is an Legendre's L.E.

Let $2x+1 = t$ $\therefore dt/dx = 2$

$$\left[t^2(2)^2 \frac{d^2y}{dt^2} - 2t(2)\frac{dy}{dt} - 12y\right] = 6\left(\frac{t-1}{2}\right)$$

$$\therefore 4t^2 \frac{d^2y}{dt^2} - 4t\frac{dy}{dt} - 12y = 6\frac{(t-1)}{2}$$

$\div 2$ $$2t^2 \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} - 6y = 3\frac{(t-1)}{2}$$

Is an homogeneous eq.

Let $t = e^z$ $\therefore z = \log t$; $D = d/dz$

$$[2D(D-1) - 2D - 6]y = \frac{3}{2}(e^z - 1)$$

$$[2D^2 - 4D - 6]y = \frac{3}{2}(e^z - 1)$$

A.E. is $D^2 - 2D - 3 = 0$ $\therefore D = 3, -1$

$\therefore y_1 = c_1 e^{3z} + c_2 e^{-z}$

$y_p = \dfrac{1}{2[(D-3)(D+1)]} \dfrac{3}{2}(e^{z} - e^{0})$

$= \dfrac{3}{2\times 2} \dfrac{1}{(-4)} e^{z} - \dfrac{3}{2\times 2} \cdot \dfrac{1}{(-3)}$

$= \left[\dfrac{3e^{z}}{-8} + \dfrac{1}{2}\right] \times \dfrac{1}{2}$

G.S. is $y = y_c + y_p$

$y = c_1 e^{3z} + c_2 e^{-z} - \dfrac{3e^{z}}{8\times 2} + \dfrac{1}{2\times 2}$

$= c_1 t^3 + c_2 t^{-1} - \dfrac{3t}{8\times 2} + \dfrac{1}{2\times 2}$

$= c_1 (2x+1)^3 + c_2 (2x+1)^{-1} - \dfrac{3(2x+1)}{2\times 8} + \dfrac{1}{2\times 2}$

$= c_1 (2x+1)^3 + c_2 (2x+1)^{-1} - \dfrac{6x}{2\times 8} - \dfrac{3}{2\times 8} + \dfrac{4}{8\times 2}$

$= c_1 (2x+1)^3 + c_2 (2x+1)^{-1} - \dfrac{3x}{8} + \dfrac{1}{16}$

5. $(x+2)^2 \dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x+4$

Let $x+2 = t$ $\therefore dx/dt = 1$

$t^2 \dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + y = 3(t-2)+4$
$= 3t-2$

Is an ~~Legen~~ Homogeneous Eq.

Let $t = e^{z}$ $\therefore z = \log t$ $\therefore D = d/dz$

$\therefore [D(D-1) - D + 1]\, y = 3e^{z} - 2$

$(D^2 - 2D + 1)\, y = 3e^{z} - 2$

A.E. is $D^2 - 2D + 1 = 0$ $\therefore$ ~~$D = \pm i$~~

~~$\therefore y_c = c_1 \cos z + c_2 \sin z$~~

$D = \dfrac{2 \pm \sqrt{4-4}}{2} = 1, 1$

$y_c = (c_1 z + c_2) e^{z}$

$y_c = (c_1 \log t + c_2)\, t = [c_1 \log(x+2) + c_2](x+2)$

$y_p = \dfrac{1}{(D-1)^2}(3e^{z} - 2e^{0})$

$= 3 \cdot \dfrac{z^2}{2!} e^{z} - \dfrac{2}{(-1)^2} = \dfrac{3}{2}\log^2(x+2) \cdot (x+2) - 2.$

$$y = y_c + y_p = (x+2)\left[c_1 + c_2 \log(x+2)\right] + \frac{3}{2}(x+2)\log^2(x+2) - 2$$